Degeneration of moduli spaces and generalized theta functions

Xiaotao Sun

Institute of Mathematics, CAS

Daejeon, August 6, 2014

Moduli spaces of semi-stable bundles

- Let C be a smooth projective curve of genus g,

$$\mathcal{U}_C = \{\text{s.s. bundles } E \text{ of } \mathrm{rk}(E) = r,\ \deg(E) = d \text{ on } C\}$$

- Let $\mathbf{Q}$ be quotient scheme of quotients E, and

$$V \otimes \mathcal{O}_{C\times\mathbf{Q}}(-N) \to \mathcal{F} \to 0, \quad \text{where } V = \mathbb{C}^{P(N)}$$

is the universal quotient on $C \times \mathbf{Q}$, $\mathrm{SL}(V)$-equivariant embeding

$$\mathbf{Q} \hookrightarrow \mathbf{G} = \mathrm{Grass}_{P(m)}(V \otimes \mathrm{H}^0(\mathcal{O}_C(m-N)))$$

- $\mathcal{U}_C$ is the GIT quotient $\mathbf{Q}^{ss} \to \mathbf{Q}^{ss}//\mathrm{SL}(V) := \mathcal{U}_C$, and

$$\Theta_{\mathbf{Q}^{ss}} := \det R\pi_{\mathbf{Q}^{ss}}(\mathcal{F})^{-k} \otimes \det(\mathcal{F}_y)^{\frac{k\chi}{r}}$$

descends to an ample line bundle $\Theta_{\mathcal{U}_C}$ on $\mathcal{U}_C$ when $r|k\chi$.

Generalized theta functions

- when $r = 1$, $H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ is space of **theta functions of order** k

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = k^g$$

- when $r > 1$, $H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C})$ is the space of so called **generalized theta functions of order** k, $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = ?$

- $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{r}\right)^g \dim H^0(\mathcal{SU}_C, \Theta_{\mathcal{U}_C})$

- A formula was predicted by **Conformal Field Theory**, when $r = 2$,

$$\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \left(\frac{k}{2}\right)^g \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{(-1)^{id}}{(\sin\frac{(i+1)\pi}{k+2})^{2g-2}}$$

Degeneration method: the case of $r = 1$

- Degenerate C to an irreducible curve X with exactly one node $x_0 \in X$, then $\mathcal{U}_C = J_C^d$ degenerates to $\mathcal{U}_X =$

$$\{\text{torsion free sheaves } E \text{ of } \mathrm{rk}(E) = 1,\ \deg(E) = d \text{ on } X\} = J_X^d$$

- One need to show: $\dim H^0(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = \dim H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X})$ ($\Leftarrow$ by $H^1(\mathcal{U}_C, \Theta_{\mathcal{U}_C}) = 0$ and $H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = 0$).

- Let $\pi : \widetilde{X} \to X$, $\pi^{-1}(x_0) = \{x_1, x_2\}$, $\mathcal{E}/\widetilde{X} \times \mathcal{U}_{\widetilde{X}}$ be a universal (line) bundle. Let $\mathcal{P} = Grass_1(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2})$, consider diagram

$$\mathcal{U}_{\widetilde{X}} \xleftarrow{\rho} \mathcal{P} \xrightarrow{\phi} \mathcal{U}_X$$

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) = \bigoplus_{i=0}^{k-1} H^0(\mathcal{U}_{\widetilde{X}}, \Theta_{\mathcal{U}_{\widetilde{X}}} \otimes \mathcal{E}_y^{-k} \otimes \mathcal{E}_{x_1}^{i} \otimes \mathcal{E}_{x_2}^{k-i})$$

Factorization Theorem: Parabolic sheaves

- We say: E has a parabolic structure of type $\vec{n}(x)$ and weights $\vec{a}(x)$ at a smooth point $x \in X$, we mean a choice of flag of quotients

$$E_x = Q_{l_x+1}(E)_x \twoheadrightarrow \cdots\cdots \twoheadrightarrow Q_1(E)_x \twoheadrightarrow Q_0(E)_x = 0$$

of fibre E_x with $n_i(x) = \dim(ker\{Q_i(E)_x \twoheadrightarrow Q_{i-1}(E)_x\})$ and a sequence of integers $0 \le a_1(x) < a_2(x) < \cdots < a_{l_x+1}(x) < k$,

$$\vec{n}(x) := (n_1(x), n_2(x), \cdots, n_{l_x+1}(x))$$

$$\vec{a}(x) := (a_1(x), a_2(x), \cdots, a_{l_x+1}(x))$$

- For any $F \subset E$, let $Q_i(E)_x^F \subset Q_i(E)_x$ be the image of F,

$$n_i^F = \dim(ker\{Q_i(E)_x^F \twoheadrightarrow Q_{i-1}(E)_x^F\})$$

$$\mathrm{par}\chi(F) := \chi(F) + \frac{1}{k}\sum_{x \in I}\sum_{i=1}^{l_x+1} a_i(x) n_i^F(x).$$

Factorization Theorem: Moduli spaces

- E is called **semistable** (resp., **stable**) for $\frac{\vec{a}}{k}$ if for any nontrivial subsheaf $E' \subset E$ such that E/E' is torsion free, one has

$$\mathrm{par}\chi(E') \leq \frac{\mathrm{par}\chi(E)}{r} \cdot r(E') \text{ (resp., } <).$$

Theorem 1 (Sun, 2000)

There exists a seminormal projective variety

$$\mathcal{U}_X := \mathcal{U}_X(r, d, I, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, k),$$

which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E of rank r and $\deg(E) = d$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at points $\{x\}_{x\in I}$. If X is smooth, then it is normal, with only rational singularities.

Factorization Theorem: Construction of moduli spaces

- $$V\otimes\mathcal{O}_{X\times\mathbf{Q}}(-N)\to\mathcal{F}\to 0,\quad \mathcal{R}=\underset{x\in I}{\times_{\mathbf{Q}}}Flag_{\vec{n}(x)}(\mathcal{F}_x)\to\mathbf{Q}$$
- Let $W_m=\mathrm{H}^0(\mathcal{O}_X(m-N))$, we have $\mathrm{SL}(V)$-equivariant embedding

$$\mathcal{R}\hookrightarrow Grass_{P(m)}(V\otimes W_m)\times\prod_{x\in I}\prod_{i=1}^{l_x}Grass_{r_i(x)}(V\otimes W_m)$$

where $r_i(x)=\mathrm{rk}(\mathcal{Q}_{\{x\}\times\mathcal{R},\,i})$. The moduli space $\mathcal{U}_X$ is GIT quotient

$$\psi:\mathcal{R}^{ss}\to\mathcal{U}_X=\mathcal{R}^{ss}/\mathrm{SL}(V)$$

under the polarzation $\frac{\ell+kN}{m-N}\times\prod_{x\in I}\{d_1(x),\cdots,d_{l_x}(x)\}$, where

$$\ell=\frac{1}{r}\left(k\chi-\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)\right)$$

and $d_i(x)=a_{i+1}(x)-a_i(x)$.

Factorization Theorem: The theta line bundles

Let $V \otimes \mathcal{O}_{X\times\mathcal{R}}(-N) \to \mathcal{E} \to 0$ be the pullback of universal quotient,

$$\mathcal{E}_{\{x\}\times\mathcal{R}} \twoheadrightarrow \mathcal{Q}_{\{x\}\times\mathcal{R},l_x} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{Q}_{\{x\}\times\mathcal{R},2} \twoheadrightarrow \mathcal{Q}_{\{x\}\times\mathcal{R},1} \twoheadrightarrow 0$$

be the universal flags of quotients. Fixed a smooth point $y \in X$, when

$$\ell = \frac{k\chi - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)r_i(x)}{r}$$

is an integer, for any integers $\{\alpha_x\}_{x\in I}$, ℓ_y with $\ell_y + \sum_{x\in I}\alpha_x = \ell$,

$$(\det R\pi_{\mathcal{R}^{ss}}\mathcal{E})^{-k} \otimes \bigotimes_{x\in I}\{\bigotimes_{i=1}^{l_x}\det(\mathcal{E}_x)^{\alpha_x} \otimes \det(\mathcal{Q}_{\{x\}\times\mathcal{R}^{ss},i})^{d_i(x)}\} \otimes \det(\mathcal{E}_y)^{\ell_y}$$

descends to an ample line bundle (theta line bundle):

$$\Theta_{\mathcal{U}_X} = \Theta(k, r, d, I, \{\vec{a}(x), \vec{n}(x), \alpha_x\}_{x\in I}, \ell_y)$$

Factorization Theorem: When X is irreducible

Theorem 2 (Sun, 2000)

$$H^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{\widetilde{X}}, \Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 \le k-1$.

- where $\pi : \widetilde{X} \to X$ is the normalization, $\pi^{-1}(x_0) = \{x_1, x_2\}$,

$$\mathcal{U}_X := \mathcal{U}_X(k, r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I}),$$
$$\Theta_{\mathcal{U}_X} := \Theta(k, r, d, \{\vec{a}(x), \vec{n}(x), \alpha_x\}_{x \in I}, \ell_y)$$

- For $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \le \mu_r \le \cdots \le \mu_1 < k$, let

$$\mathcal{U}^{\mu}_{\widetilde{X}} := \mathcal{U}_{\widetilde{X}}(k, r, d, \{\vec{n}(x), \vec{a}(x)\}_{x \in I \cup \{x_1, x_2\}}),$$
$$\Theta_{\mathcal{U}^{\mu}_{\widetilde{X}}} := \Theta(k, r, d, \{\vec{a}(x), \vec{n}(x), \alpha_x\}_{x \in I \cup \{x_1, x_2\}}, \ell_y)$$

Remarks of Factorization Theorem

- For $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \le \mu_r \le \cdots \le \mu_1 < k$, let

$$\{d_i = \mu_{r_i} - \mu_{r_i+1}\}_{1\le i\le l}$$

be the subset of nonzero integers in $\{\mu_i - \mu_{i+1}\}_{i=1,\cdots,r-1}$.

- Define $r_i(x_1) = r_i, \quad d_i(x_1) = d_i, \quad l_{x_1} = l, \quad \alpha_{x_1} = \mu_r$ and

$$r_i(x_2) = r - r_{l-i+1}, \quad d_i(x_2) = d_{l-i+1}, \quad l_{x_2} = l, \quad \alpha_{x_2} = k - \mu_1$$

$$\vec{a}(x_j) = \left(\mu_r, \mu_r + d_1(x_j), \cdots, \mu_r + \sum_{i=1}^{l_{x_j}-1} d_i(x_j), \mu_r + \sum_{i=1}^{l_{x_j}} d_i(x_j)\right)$$

$$\vec{n}(x_j) = (r_1(x_j), r_2(x_j) - r_1(x_j), \cdots, r_{l_{x_j}}(x_j) - r_{l_{x_j}-1}(x_j)).$$

- When $r = 2$, it is due to Narasimhan-Ramadas.

Factorization Theorem: When $X = X_1 \cup X_2$ reducible

Theorem 3 (Sun, 2003)

$$H^0(\mathcal{U}_{X_1\cup X_2}, \Theta_{\mathcal{U}_{X_1\cup X_2}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{X_1}, \Theta_{\mathcal{U}^{\mu}_{X_1}}) \otimes H^0(\mathcal{U}^{\mu}_{X_2}, \Theta_{\mathcal{U}^{\mu}_{X_2}})$$

where $\mu = (\mu_1, \cdots, \mu_r)$ *runs through* $0 \le \mu_r \le \cdots \le \mu_1 < k$.

- where $\pi : X_1 \sqcup X_2 \to X_1 \cup X_2$, $I = I_1 \cup I_2$, and $\ell = \ell_1 + \ell_2$,

$$\mathcal{U}_{X_1\cup X_2} := \mathcal{U}_X(r, d, I_1 \cup I_2, \{\vec{n}(x), \vec{a}(x)\}_{x\in I}, \mathcal{O}(1), k)$$

  choose $\mathcal{O}(1) = \mathcal{O}_X(c_1y_1 + c_2y_2)$ such that $\ell_i = \frac{c_i\ell}{c_1+c_2}$ are integers.

- $$\Theta_{\mathcal{U}_{X_1\cup X_2}} = \Theta(k, r, d, I_1 \cup I_2, \{\vec{a}(x), \vec{n}(x), \alpha_x\}_{x\in I_1\cup I_2}, \ell_{y_1}, \ell_{y_2}),$$

$$\ell_{y_i} + \sum_{x\in I_i} \alpha_x = \ell_i \quad (i = 1, 2).$$

Factorization Theorem: Notation

- For $\mu = (\mu_1, \cdots, \mu_r)$ with $0 \le \mu_r \le \cdots \le \mu_1 < k$, we define

$$d_1^{\mu} = \frac{1}{k}\left(\sum_{x\in I_1}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\ell_1\right) + r(g_1 - 1) + \frac{1}{k}\sum_{i=1}^{r}\mu_i$$

$$d_2^{\mu} = \frac{1}{k}\left(\sum_{x\in I_2}\sum_{i=1}^{l_x} d_i(x)r_i(x) + r\ell_2\right) + r(g_2 - 1) + r - \frac{1}{k}\sum_{i=1}^{r}\mu_i$$

- For $j = 1, 2$, we define

$$\mathcal{U}^{\mu}_{X_j} := \mathcal{U}_{X_j}(r, d^{\mu}_j, I_j \cup \{x_j\}, \{\vec{n}(x), \vec{a}(x)\}_{x\in I_j\cup\{x_j\}}, k),$$

$$\Theta_{\mathcal{U}^{\mu}_{X_j}} = \Theta(k, r, d^{\mu}_j, I_j \cup \{x_j\}, \{\vec{n}(x), \vec{a}(x), \alpha_x\}_{x\in I_j\cup\{x_j\}}, \ell_{y_j})$$

Vanishing Theorem: The case of smooth curves

- For any data $\omega = (k, r, d, \{\vec{n}(x), \vec{a}(x)\}_{x\in I})$ such that

$$\ell := \frac{k(d + r(1-g)) - \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x)}{r}$$

is an integer, we have the moduli space $\mathcal{U}_{X,\omega} = \mathcal{U}_X(\omega)$ and

$$\Theta_{\mathcal{U}_{X,\omega}} = \Theta(k, r, d, \{\vec{n}(x), \vec{a}(x), \alpha_x\}_{x\in I}, \ell_y)$$

where $\ell = \ell_y + \sum_x \alpha_x$.

Theorem 4 (Sun, 2000)

Let X be a smooth projective curve of genus g. Then

$$\mathrm{H}^1(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\omega}}) = 0$$

if $(r-1)(g-1) + \frac{|I|}{k+2r} \geq 2$.

Vanishing Theorem: The case of singular curves

Theorem 5 (Sun, 2000)

Let X *be an irreducible projective curve of genus* g *with one node and* $(r-1)(g-2)+\frac{|I|}{k+2r}\geq 2$*. Then*

$$H^1(\mathcal{U}_X,\Theta_{\mathcal{U}_X,\omega})=0.$$

Theorem 6 (Sun, 2013)

Let $X=X_1\cup X_2$ *be an reducible projective curve of genus* g *with one node and* $(r-1)g+\frac{|I|}{k+2r}\geq 2$*. Then*

$$H^1(\mathcal{U}_X,\Theta_{\mathcal{U}_X,\omega})=0.$$

Proof of Vanishing Theorems: When X is smooth

Theorem 7 (Sun, 2000)

For any $\omega = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I})$, *we have*

(1) $\mathrm{codim}(\mathcal{R}^{ss}_{\omega} \setminus \mathcal{R}^{s}_{\omega}) \geq (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$,

(2) $\mathrm{codim}(\mathcal{R} \setminus \mathcal{R}^{ss}_{\omega}) > (r-1)(g-1) + \frac{|\mathrm{I}|}{k}$.

Theorem 8 (Sun, 2000)

Let $\omega_X = \mathcal{O}_X(\sum q)$, $\omega_{\mathcal{R}}$ *be the canonical sheaf of* X, $\mathcal{R}$. *Then*

$$\begin{aligned}\omega_{\mathcal{R}}^{-1} =& (det\, R\pi_{\mathcal{R}}\mathcal{F})^{-2r} \otimes \\ & \bigotimes_{x\in I}\left\{(det\,\mathcal{F}_x)^{n_{l_x+1}-r} \otimes \bigotimes_{i=1}^{l_x}(det\,\mathcal{Q}_{x,i})^{n_i(x)+n_{i+1}(x)}\right\} \\ & \otimes \bigotimes_{q}(det\,\mathcal{F}_q)^{1-r} \otimes (det\, R\pi_{\mathcal{R}} det\mathcal{F})^2.\end{aligned}$$

Proof of Vanishing Theorems: When X is smooth

- For any $\omega = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I})$, we have

$$\begin{aligned} \mathrm{H}^1(\mathcal{U}_{X,\,\omega}, \Theta_{\mathcal{U}_{X,\,\omega}}) &= \mathrm{H}^1(\mathcal{R}^{ss}_{\omega}, \hat{\Theta}_{\mathcal{R}})^{inv} = \mathrm{H}^1(\mathcal{R}, \hat{\Theta}_{\mathcal{R}})^{inv} \\ &= \mathrm{H}^1(\mathcal{R}, \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2} \otimes \omega_{\mathcal{R}})^{inv} \\ &= \mathrm{H}^1(\mathcal{U}_{X,\,\bar{\omega}}, \Theta_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2} \otimes \omega_{\mathcal{U}_{X,\,\bar{\omega}}}) = 0 \end{aligned}$$

where $\hat{\Theta}_{\mathcal{R}} \otimes \omega_{\mathcal{R}}^{-1} = \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2}$, $\hat{\Theta}_{\bar{\omega}}$ is determined by

$$\bar{\omega} = (\bar{k}, r, d, \{\vec{n}(x), \{\bar{d}_i(x)\}_{1\le i\le l_x}\}_{x\in I}), \quad \bar{k} = k + 2r, \cdots$$

- $\mathrm{Det} : \mathcal{R} \to J^d_X$ induces $\mathrm{Det} : \mathcal{U}_X \to J^d_X$, one can prove that

$$\Theta_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2}$$

is ample

Remarks about the proof: When X is smooth

Theorem 9 (Sun, 2013)

For any data $\omega = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I})$, the dimension of

$$\mathrm{H}^0(\mathcal{U}_{X,\,\omega}, \Theta_{\mathcal{U}_{X,\,\omega}})$$

is independent of the choices of curve X and the points $x \in X$.

- For any data $\omega = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I})$, we choose

  $$\omega(J) = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I\cup J})$$

  such that $(r-1)(g-1) + \frac{|I\cup J|}{k+2r} \ge 2$.

- The projection $p_I : \widetilde{\mathcal{R}} = \underset{x\in I\cup J}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x) \to \mathcal{R} = \underset{x\in I}{\times_{\mathbf{Q}}} Flag_{\vec{n}(x)}(\mathcal{F}_x)$ is $\mathrm{SL}(V)$-invariant.

Remarks about the proof: When X is smooth

- $\mathrm{H}^0(\mathcal{U}_{X,\,\omega}, \Theta_{\mathcal{U}_{X,\,\omega}}) = \mathrm{H}^0(\mathcal{R}^{ss}_{\omega}, \hat{\Theta}_{\mathcal{R}})^{inv} = \mathrm{H}^0(\mathcal{R}, \hat{\Theta}_{\mathcal{R}})^{inv} = \mathrm{H}^0(\widetilde{\mathcal{R}}, p_I^*(\hat{\Theta}_{\mathcal{R}}))^{inv}$

- $p_I^*(\hat{\Theta}_{\mathcal{R}}) = \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2} \otimes \omega_{\widetilde{\mathcal{R}}}$, where
$$p_I^*(\hat{\Theta}_{\mathcal{R}}) \otimes \omega^{-1}_{\widetilde{\mathcal{R}}} = \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2},$$
$\hat{\Theta}_{\bar{\omega}}$ is determined by
$$\bar{\omega} = (\bar{k}, r, d, \{\vec{n}(x), \{\bar{d}_i(x)\}_{1\le i\le l_x}\}_{x\in I\cup J}), \quad \bar{k} = k + 2r, \cdots$$

- $\mathrm{H}^0(\mathcal{R}^{ss}_{\omega}, \hat{\Theta}_{\mathcal{R}})^{inv} = \mathrm{H}^0(\widetilde{\mathcal{R}}, \hat{\Theta}_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2} \otimes \omega_{\widetilde{\mathcal{R}}})^{inv}$
$$\psi : \widetilde{\mathcal{R}}^{ss}_{\bar{\omega}} \to \mathcal{U}_{X,\,\bar{\omega}}$$

- (Knnop): $(\psi_*\omega_{\widetilde{\mathcal{R}}})^{inv} = \omega_{\mathcal{U}_{X,\,\bar{\omega}}}$ if $codim(\widetilde{\mathcal{R}}^{ss}_{\bar{\omega}} \setminus \widetilde{\mathcal{R}}^{s}_{\bar{\omega}}) \ge 2$.

- $\mathrm{H}^0(\mathcal{U}_{X,\,\omega}, \Theta_{\mathcal{U}_{X,\,\omega}}) = \mathrm{H}^0(\mathcal{U}_{X,\,\bar{\omega}}, \Theta_{\bar{\omega}} \otimes \mathrm{Det}^*(\Theta_y)^{-2} \otimes \omega_{\mathcal{U}_{X,\,\bar{\omega}}})$

Sketch of Proof: Normalization $\phi : \mathcal{P} \to \mathcal{U}_X$ of $\mathcal{U}_X$

- Let $\pi : \widetilde{X} \to X$, $\pi^{-1}(x_0) = \{x_1, x_2\}$, $\widetilde{V} \otimes \mathcal{O}_{\widetilde{X}\times\widetilde{\mathcal{R}}}(-N) \to \widetilde{\mathcal{F}} \to 0$

$$\rho : \widetilde{\mathcal{R}}' = \mathrm{Grass}_r(\widetilde{\mathcal{F}}_{x_1} \oplus \widetilde{\mathcal{F}}_{x_2}) \to \widetilde{\mathcal{R}}$$

- $\mathcal{P} = \{$ semi-stable GPS $(E, Q) = (E, E_{x_1} \oplus E_{x_2} \to Q \to 0) \}$, which is called **moduli space of GPS** (*generalized parabolic sheaf*), and

$$\widetilde{\mathcal{R}}'^{ss} \to \mathcal{P} := \widetilde{\mathcal{R}}'^{ss} // \mathrm{SL}(\widetilde{V})$$

- $\phi : \mathcal{P} \to \mathcal{U}_X$ is defined by $\phi(E, Q) = F$, where F is given by

$$0 \to F \to \pi_* E \to_{x_0} Q \to 0$$

- $\phi^* \Theta_{\mathcal{U}_X} = \Theta_{\mathcal{P}} = \rho^*(\Theta_{\mathcal{U}_{\widetilde{X}}}) \otimes \eta^k_{x_2}$, where $\eta_{x_2} = \det(\mathcal{Q}) \otimes \det(\mathcal{E}_{x_2})^{-1}$

Sketch of Proof: Filtration and Singularities

- $$\mathcal{P} \supset \mathcal{D}_i := \mathcal{D}_i(r-1) \supset \cdots \supset \mathcal{D}_i(a) \supset \mathcal{D}_i(a-1) \supset \cdots \supset \mathcal{D}_i(0),$$
  $$\mathcal{D}_i(a) = \{\, (E, Q) \,|\, \mathrm{rank}(E_{x_i} \to Q) \leq a \,\}, \ (i = 1, 2)$$

- $$\mathcal{U}_X \supset \mathcal{W}_{r-1} \supset \cdots \supset \mathcal{W}_a \supset \mathcal{W}_{a-1} \supset \cdots \supset \mathcal{W}_0, \text{ where}$$
  $$\mathcal{W}_a := \{\, [F] \in \mathcal{U}_X \,|\, F \otimes \hat{\mathcal{O}}_{x_0} = \hat{\mathcal{O}}_{x_0}^{\oplus t} \oplus \hat{m}_{x_0}^{\oplus (r-t)}, \ t \leq a \,\}$$

Theorem 10 (Sun, 2000)

- *$\mathcal{U}_X$ and $\mathcal{W}_a$ $(0 \leq a < r)$ have only seminormal singularities.*
- *$\mathcal{P}$ and $\mathcal{D}_i(a)$ $(0 \leq a < r)$ are normal with only rational singularities.*
- *$\phi|_{\mathcal{D}_1(a)} : \mathcal{D}_1(a) \to \mathcal{W}_a$ is the normalization of $\mathcal{W}_a$, and*
  $$\phi : \mathcal{D}_1(a) \setminus \{\mathcal{D}_1(a) \cap \mathcal{D}_2 \cup \mathcal{D}_1(a-1)\} \cong \mathcal{W}_a \setminus \mathcal{W}_{a-1}$$

Sketch of Proof: Factorization

- For any $i > 0$, $\quad \phi^* : \mathrm{H}^i(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \mathrm{H}^i(\mathcal{P}, \Theta_{\mathcal{P}})$.

- $\mathrm{H}^0(\mathcal{U}_X, \Theta_{\mathcal{U}_X}) \cong \mathrm{H}^0(\mathcal{P}, \Theta_{\mathcal{P}}(-\mathcal{D}_1)) = \mathrm{H}^0(\mathcal{U}_{\widetilde{X}}, \Theta_{\mathcal{U}_{\widetilde{X}}} \otimes \rho_* \eta_{x_2}^{k-1})$.

- $\forall\ [E] \in \mathcal{U}_{\widetilde{X}}, \quad \rho^{-1}([E]) = Grass_r(E_{x_1} \oplus E_{x_2}) := Gr,$

$$\eta_{x_2} = \det(Q) \otimes \det(E_{x_2})^{-1}$$

$$\mathrm{H}^0(Gr, \eta_{x_2}^{k-1}) = \bigoplus_{\mu} \mathbb{S}_\mu(E_{x_1}) \otimes \mathbb{S}_\mu(E_{x_2}^*)$$

where $\mu = (\mu_1, \cdots, \mu_r)$ runs through $0 \le \mu_r \le \cdots \le \mu_1 < k$.

- $\mathbb{S}_\mu(E_{x_1}) \otimes \mathbb{S}_\mu(E_{x_2}^*) = \mathrm{H}^0(\mathcal{Z}_1^\mu \times \mathcal{Z}_2^\mu, p_1^*\mathcal{L}_1^\mu \otimes p_2^*\mathcal{L}_2^\mu)$, where

$$\mathcal{Z}_i^\mu := Flag_{\vec{n}(x_i)}(E_{x_i}),\ (i = 1, 2)$$

Proof of Vanishing Theorems: When X has one node

- For any $\omega = (k, r, d, \{\vec{n}(x), \{d_i(x)\}_{1\le i\le l_x}\}_{x\in I})$, we have

$$H^1(\mathcal{U}_X, \Theta_{\mathcal{U}_X,\omega}) \cong H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P},\omega}).$$

- We have also $\mathrm{Det} : \mathcal{P} \to J^d_{\widetilde{X}}$, and

$$H^1(\mathcal{P}_\omega, \Theta_{\mathcal{P},\omega}) = H^1(\mathcal{P}_{\bar\omega}, \Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*(\Theta_J^{-1}) \otimes \omega_{\mathcal{P}_{\bar\omega}})$$

- When X is irreducible, $\Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*(\Theta_J^{-1})$ may not be ample, then it was reduced to the case of fixed determinant (where $\mathrm{Det}^*(\Theta_J^{-1})$ disappear).

- When $X = X_1 \cup X_2$, $\Theta_{\mathcal{P},\bar\omega} \otimes \mathrm{Det}^*(\Theta_J^{-1})$ is indeed ample.

Frobenius splitting and Frobenius splitting type of varieties

- A variety M defined over a prefect field k of $char(k) = p > 0$ is called Frobenius splitting if
$$0 \to \mathcal{O}_M \to F_*\mathcal{O}_M \to B_M^1 \to 0$$
is splitting, where $F : M \to M$ is the Frobenius morphism of M.

- A variety M defined over a field of characteristic zero is called of Frobenius splitting type if its modulo p reduction is Frobenius splitting for almost p.

- Let $f : M' \to M$ be a morphism such that $f_*(\mathcal{O}_{M'}) = \mathcal{O}_M$ and M' be Frobenius splitting (resp. of Frobenius splitting type). Then so is M.

- A normal proper Fano variety with only rational singularities is of Frobenius splitting type (Smith).

Theorem 11 (Sun-Zhou, 2014)

Let X be a smooth projective curve. Then, for any data ω, the moduli spaces $\mathcal{U}_{X,\omega}^L$ and $\mathcal{P}_{X,\omega}^L$ of parabolic and generalized parabolic sheaves with fixed determinant L are of Frobenius splitting type.

- Recall $p_I : \widetilde{\mathcal{R}}_L \to \mathcal{R}_L$, there are data $\widetilde{\omega}$ such that
$$\mathcal{U}_{X,\widetilde{\omega}}^L = \widetilde{\mathcal{R}}_{L,\widetilde{\omega}}^{ss} // SL(V)$$
is a normal, proper Fano variety with only rational singularities.
- Let $\widetilde{U} = p_I^{-1}(\mathcal{R}_{L,\omega}^{ss}) \cap \widetilde{\mathcal{R}}_{L,\widetilde{\omega}}^s$, then $codim(\widetilde{\mathcal{R}}_{L,\widetilde{\omega}}^{ss} \setminus \widetilde{U}) \geq 2$.
- Let $U \subset \mathcal{U}_{X,\widetilde{\omega}}^L$ be the image of $\widetilde{U}$, then p_I induces a morphism
$$f : U \to \mathcal{U}_{X,\omega}^L$$
such that $f_*(\mathcal{O}_U) = \mathcal{O}_{\mathcal{U}_{X,\omega}^L}$.

Vanishing theorems

Corollary 1

Let X be a projective curve with at most one node and $\mathcal{U}_{X,\omega}$ be the moduli space of parabolic sheaves on X with any given data ω. Then

$$H^i(\mathcal{U}_{X,\omega}, \Theta_{\mathcal{U}_{X,\,\omega}}) = 0, \quad \forall \ i > 0$$

Thanks !